THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH2230A Complex Variables with Applications 2017-2018
Suggested Solution to Assignment 10

§72) 3) For any z with $\frac{|z-2|}{2} < 1$,

$$\frac{1}{z} = \frac{1}{2} \cdot \frac{1}{1-(-\frac{z-2}{2})} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \left(\frac{z-2}{2}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{(z-2)^n}{2^{n+1}}$$

By differentiating the series term by term, we have

$$\begin{aligned} -\frac{1}{z^2} &= \sum_{n=1}^{\infty}(-1)^n (n) \frac{(z-2)^{n-1}}{2^{n+1}} \\ \frac{1}{z^2} &= -\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n (n) \frac{(z-2)^{n-1}}{2^{n-1}} \\ \frac{1}{z^2} &= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n (n+1) \frac{(z-2)^{n}}{2^{n}} \end{aligned}$$

§72) 5) For $z \neq \pi/2$,

$$\begin{aligned} f(z) &= \frac{\cos z}{z^2 - (\pi/2)^2} \\ &= \frac{-1}{(z+\pi/2)(z-\pi/2)} \sin(z-\pi/2) \\ &= \frac{-1}{(z+\pi/2)(z-\pi/2)} \sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi/2)^{2n+1}}{(2n+1)!} \\ &= \frac{-1}{(z+\pi/2)} \sum_{n=0}^{\infty} \frac{(-1)^n (z-\pi/2)^{2n}}{(2n+1)!}, \end{aligned}$$

where the last expression is also well-defined at $z = \pi/2$ with value $\dfrac{-1}{\pi}$.

Similarly, for $z \neq -\pi/2$,

$$\begin{aligned} f(z) &= \frac{\cos z}{z^2 - (\pi/2)^2} \\ &= \frac{1}{(z+\pi/2)(z-\pi/2)} \sin(z+\pi/2) \\ &= \frac{1}{(z+\pi/2)(z-\pi/2)} \sum_{n=0}^{\infty} \frac{(-1)^n (z+\pi/2)^{2n+1}}{(2n+1)!} \\ &= \frac{1}{(z-\pi/2)} \sum_{n=0}^{\infty} \frac{(-1)^n (z+\pi/2)^{2n}}{(2n+1)!}, \end{aligned}$$

where the last expression is also well-defined at $z = -\pi/2$ with value $\dfrac{-1}{\pi}$.

Therefore, $z = \pm\pi/2$ are removable singularities and $f(z)$ is an entire function.

§73) 1) For $0 < |z| < 1$,

$$\begin{aligned}
\frac{e^z}{z(z^2+1)} &= \left(\frac{1}{z}\right)\left(\sum_{n=0}^{\infty}\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n z^{2n}\right) \\
&= \left(\frac{1}{z}\right)\left[(1)(1) + (z)(1) + [\frac{z^2}{2}(1) + (1)(-z^2)] + [(z)(-z^2) + \frac{z^3}{3!}(1)] + \dots\right] \\
&= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \dots
\end{aligned}$$

§73) 4) Since $e^z - 1 = \displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n!}$, the division is given by

$$\begin{array}{rl}
 & \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{z^3}{720} + \dots \\ \hline
z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \dots \;\big) & 1 \\
 & 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \dots \\ \hline
 & -\frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} - \frac{z^4}{120} + \dots \\
 & -\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{12} - \frac{z^4}{48} + \dots \\ \hline
 & \frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{80} + \dots \\
 & \frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{72} + \dots \\ \hline
 & -\frac{z^4}{720} + \dots \\
 & -\frac{z^4}{720} + \dots \\ \hline
\end{array}$$

§77) 1) b) For $0 < |z| < \infty$, $z\cos\left(\frac{1}{z}\right) = z\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!z^{2n}} = z - \frac{1}{2z} + \frac{1}{24z^3} + \dots$

Hence the residue at $z = 0$ is $-\frac{1}{2}$.

§77) 2) b) Note that for $z \neq 1$, $\dfrac{e^{-z}}{(z-1)^2} = e^{-1}\dfrac{e^{-(z-1)}}{(z-1)^2} = e^{-1}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^{n-2}}{n!}$ In particular, the coefficient of z^{-1} of the series expansion is $-e^{-1}$. Hence, by Residue Theorem, we have

$$\int_{|z|=3}\frac{e^{-z}}{(z-1)^2}dz = 2\pi i(-e^{-1}) = -2\pi i/e$$

§77) 4) a) Let $f(z) = \dfrac{z^5}{1-z^3}$. Then we have for $0 < |z| < 1$,

$$\frac{1}{z^2}f\left(\frac{1}{z}\right) = \frac{1}{z^7 - z^4} = \frac{-1}{z^4}\cdot\frac{1}{1-z^3} = \frac{-1}{z^4}\sum_{n=0}^{\infty}z^{3n} = -\sum_{n=0}^{\infty}z^{3n-4}$$

Therefore,

$$\int_{|z|=2} f(z)dz = 2\pi i\,\mathrm{Res}_{z=0}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right)\right] = -2\pi i$$

Remark: In the examination, to apply the theorem about residue at infinity, it is better to check that all the singularities of the function $f(z)$ lie inside the contour. Otherwise you may lose some marks (depending on the difficulties of the exam).

§79) 1) a) Note that for $z \neq 0$,

$$z\exp\left(\frac{1}{z}\right) = z\sum_{n=0}^{\infty}\frac{1}{n!z^n} = \sum_{n=0}^{\infty}\frac{1}{n!z^{n-1}}$$

The principle part is given by $\sum_{n=2}^{\infty} \frac{1}{n!z^{n-1}}$.

Therefore the singular point $z = 0$ is an essential singularity.

c) Note that for $z \neq 0$,

$$\frac{\sin z}{z} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n+1)!}$$

The principle part is 0.

Therefore the singular point $z = 0$ is a removable singularity.

§79) 2) c) Note that for $z \neq 1$,

$$\frac{e^{2z}}{(z-1)^2} = e^2 \frac{e^{2z-2}}{(z-1)^2} = e^2 \frac{1}{(z-1)^2} \sum_{n=0}^{\infty} \frac{2^n (z-1)^n}{n!} = \frac{e^2}{(z-1)^2} + \frac{2e^2}{z-1} + \dots$$

Therefore the singular point $z = 1$ is a pole of order 2 and the residue is $2e^2$.